A generalization of Taub-NUT deformations

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Abstract

We introduce a generalization of Taub-NUT deformations for large families of hyper-Kähler quotients including toric hyper-Kähler manifolds and quiver varieties. It is well-known that Taub-NUT deformations are defined for toric hyper-Kähler manifolds, and the similar deformations were introduced for ALE hyper-Kähler manifolds of type D_k by Dancer, using the complete hyper-Kähler metric on the cotangent bundle of complexification of compact Lie group. We generalize them and study the Taub-NUT deformations for the Hilbert schemes of k points on $\mathbb{C}^2$.

1 Introduction

1.1 Taub-NUT spaces

A hyper-Kähler manifold is a Riemannian manifold (M, g) equipped with orthogonal integrable complex structures I_1, I_2, I_3 with quaternionic relations $I_1^2 = I_2^2 = I_3^2 = I_1I_2I_3 = -1$ so that each (M, g, I_i) is Kählerian. Then M admits three symplectic forms $\omega_i := g(I_i\cdot, \cdot)$, each of which is Ricci-flat Kähler metric with respect to I_i. Throughout of this article, we regard (M, I_1) as a complex manifold with a Ricci-flat Kähler metric ω_1 and a non-degenerate closed $(2, 0)$-form $\omega_2 + \sqrt{-1}\omega_3$, so-called a holomorphic symplectic structure.

The Euclidean space $\mathbb{C}^2 = \mathbb{R}^4$ is the trivial example of complete hyper-Kähler manifold, whose Ricci-flat Kähler metric is Euclidean and the holomorphic symplectic structure is given by $dz \wedge dw$, where $(z, w) \in \mathbb{C}^2$ is the standard holomorphic coordinate.

In [7], Hawking constructed a complete hyper-Kähler metric on $\mathbb{R}^4$ with cubic volume growth which is called a Taub-NUT space. On the other hand,

LeBrun [13] showed that the Taub-NUT space and the Euclidean space $\mathbb{C}^2$ are isomorphic as holomorphic symplectic manifolds, consequently biholomorphic. It means that the complex manifold $\mathbb{C}^2$ admits at least 2 complete Ricci-flat Kähler metrics which are not isometric. Such a phenomenon should never occur on compact complex manifolds, due to the uniqueness of the Ricci-flat Kähler metrics in each Kähler class. The similar relation also holds between multi Eguchi-Hanson spaces and multi Taub-NUT spaces.

A generalization to the higher dimensional case are obtained by Gibbons, Rychenkova, Goto [6] and Bielawski [1]. They construct Taub-NUT like hyper-Kähler metrics by deforming the toric hyper-Kähler manifolds, using tri-Hamiltonian torus actions and hyper-Kähler quotient method.

In [2], Dancer has defined the analogy of Taub-NUT deformations for some of the ALE spaces of type D_k using $U(2)$-actions. His results are based on the existence of hyper-Kähler metrics on $T^*G^{\mathbb{C}}$ for any compact Lie group G constructed by Kronheimer [12]. Another generalization to noncommutative case is considered in Section 5 of [4]. They considered *hyper-Kähler modifications* for hyper-Kähler manifolds with a tri-Hamiltonian H-action, where H is a compact Lie group which is possibly noncommutative. Note that the case of [4] does not contains the results in [2], since the ALE spaces of type D_k have no nontrivial tri-Hamiltonian actions.

In this paper, we generalize Taub-NUT deformations for some kinds of hyper-Kähler quotients, which enable us to treat the above three cases [6, 1], [2] and Section 5 of [3] uniformly. As a consequence, we apply the Taub-NUT deformations for the Hilbert schemes of k-points on $\mathbb{C}^2$.

1.2 Notation and a main result

Here, we describe the main result in this paper more precisely. Let a compact connected Lie group H act on a hyper-Kähler manifold (M, g, I_1, I_2, I_3) preserving the hyper-Kähler structure and there exists a hyper-Kähler moment map $\hat{\mu} : M \to \mathrm{Im}\mathbb{H} \otimes \mathbf{h}^*$ with respect to H-action, where $\mathrm{Im}\mathbb{H} \cong \mathbb{R}^3$ be the pure imaginary part of quaternion $\mathbb{H}$ and $\mathbf{h}^*$ is the dual space of the Lie algebra $\mathbf{h} = \mathrm{Lie}(H)$. Moreover, suppose the H-action extends to holomorphic $H^{\mathbb{C}}$ action on (M, I_1), where $H^{\mathbb{C}}$ is the complexification of H. Let $\rho : H \to G \times G$ is a homomorphism of Lie groups, where G is compact connected Lie group. Then $H_\rho := \rho^{-1}(\Delta_G) \subset H$ acts on M, where $\Delta_G \subset G \times G$ is the diagonal subgroup, and the inclusion $\iota : H_\rho \to H$ induces a hyper-Kähler moment map $\mu := \iota^* \circ \hat{\mu}$. If we denote by $Z_H \subset \mathbf{h}^*$ the subspace of fixed points by coadjoint action of H on $\mathbf{h}^*$, then we have the hyper-Kähler quotient $\mu^{-1}(\iota^*\zeta)/H_\rho$ for each $\zeta \in \mathrm{Im}\mathbb{H} \otimes Z_H$. In this paper we define Taub-NUT deformations for $\mu^{-1}(\iota^*\zeta)/H_\rho$ by the following way.

Let $N_G = T^*G^{\mathbb{C}}$ be the hyper-Kähler manifolds with a $G \times G$-action constructed by Kronheimer [12], and $\nu : N_G \to \mathrm{Im}\mathbb{H} \otimes (\mathbf{g} \oplus \mathbf{g})^*$ be its hyper-Kähler moment map described by Dancer and Swann [3]. Then H acts on $M \times N_G$ by ρ, and for $(x,p) \in M \times N_G$, $\sigma(x,p) := \hat{\mu}(x) + \rho^*(\nu(p))$ becomes the hyper-Kähler moment map, accordingly we obtain a hyper-Kähler quotient $\sigma^{-1}(\zeta)/H$ for each $\zeta \in \mathrm{Im}\mathbb{H} \otimes Z_H$. Now we have two hyper-Kähler quotients $\mu^{-1}(\iota^*\zeta)/H_\rho$ and $\sigma^{-1}(\zeta)/H$. If they are smooth, then there are Ricci-flat Kähler metrics $\omega_1^{\iota^*\zeta}, \omega_1^{\zeta}$ and holomorphic symplectic structures $\omega_2^{\iota^*\zeta} + \sqrt{-1}\omega_3^{\iota^*\zeta}$, $\omega_2^{\zeta} + \sqrt{-1}\omega_3^{\zeta}$, respectively. Next we extend ρ to the holomorphic homomorphism $H^{\mathbb{C}} \to G^{\mathbb{C}} \times G^{\mathbb{C}}$ and obtain a holomorphic map

$$\bar{\rho} : H^{\mathbb{C}}/H_\rho^{\mathbb{C}} \to (G^{\mathbb{C}} \times G^{\mathbb{C}})/\Delta_{G^{\mathbb{C}}}. \tag{1}$$

Then the main result is described as follows.

Theorem 1.1. *Let $M = \mathbb{H}^N$, and $H \subset Sp(N)$ acts on M naturally. Assume $\bar{\rho}$ is surjective. Then there exists an biholomorphism as complex analytic spaces*

$$\psi : \mu^{-1}(\iota^*\zeta)/H_\rho \to \sigma^{-1}(\zeta)/H$$

for each $\zeta \in Z_H$. Moreover, if H_ρ acts on $\mu^{-1}(\iota^*\zeta)$ freely, then $\mu^{-1}(\iota^*\zeta)/H_\rho$ and $\sigma^{-1}(\zeta)/H$ are smooth complete hyper-Kähler manifolds and ψ satisfies

$$[\psi^*\omega_1^{\zeta}]_{DR} = [\omega_1^{\iota^*\zeta}]_{DR}, \quad \psi^*(\omega_2^{\zeta} + \sqrt{-1}\omega_3^{\zeta}) = \omega_2^{\iota^*\zeta} + \sqrt{-1}\omega_3^{\iota^*\zeta},$$

where $[\cdot]_{DR}$ is a de Rham cohomology class.

1.3 Hilbert schemes of k points on $\mathbb{C}^2$

We apply the above theorem to $M = \mathrm{End}(\mathbb{C}^k) \otimes_{\mathbb{C}} \mathbb{H}$, $H = U(k) \times U(k)$, $G = U(k)$ and $\rho = \mathrm{id}$. Then $Z_H = \mathbb{R}$, and $\mu^{-1}(\iota^*\zeta)/H_\rho$ becomes a quiver varieties constructed in [14], called the Hilbert scheme of k-points of $\mathbb{C}^2$. In particular, $\mu^{-1}(0)/H_\rho$ is isomorphic to $(\mathbb{C}^2)^k/\mathcal{S}_k$ with Euclidean metric as hyper-Kähler orbifolds. In this case, $\sigma^{-1}(\zeta)/H$ becomes a smooth hyper-Kähler manifolds diffeomorphic to $\mu^{-1}(\iota^*\zeta)/H_\rho$ by Theorem 1.1, and the hyper-Kähler metric on $\sigma^{-1}(0)/H$ can be described concretely.

Theorem 1.2. *In the above situation, we have an isomorphism*

$$\sigma^{-1}(0)/H \cong (\mathbb{C}^2_{Taub-NUT})^k/\mathcal{S}_k$$

as hyper-Kähler orbifolds, where $\mathbb{C}^2_{Taub-NUT}$ is the Taub-NUT space.

1.4 Outline of the proof

Theorem 1.1 is proven in the following way. The hyper-Kähler moment map μ is decomposed into $\mu = (\mu_1, \mu_{\mathbb{C}} := \mu_2 + \sqrt{-1}\mu_3)$ along the decomposition $\mathrm{Im}\mathbb{H} = \mathbb{R}\oplus\mathbb{C}$, where $\mu_1 : M \to \mathbf{h}_\rho^*$ and $\mu_{\mathbb{C}} : M \to (\mathbf{h}_\rho^{\mathbb{C}})^*$, and the other hyper-Kähler moment maps and the parameter $\zeta \in \mathrm{Im}\mathbb{H}\otimes Z_H$ are also decomposed in the same manner. Define sets of "stable points" by

$$\begin{aligned}
\mu_{\mathbb{C}}^{-1}(\iota^*\zeta_{\mathbb{C}})_{\iota^*\zeta_1} &:= H_\rho^{\mathbb{C}} \cdot \mu^{-1}(\iota^*\zeta) \\
&= \{g \cdot x \in \mu_{\mathbb{C}}^{-1}(\iota^*\zeta_{\mathbb{C}});\ g \in H_\rho^{\mathbb{C}},\ x \in \mu^{-1}(\iota^*\zeta)\}, \\
\sigma_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})_{\zeta_1} &:= H^{\mathbb{C}} \cdot \sigma^{-1}(\zeta).
\end{aligned}$$

The natural embedding

$$\mu^{-1}(\iota^*\zeta) \hookrightarrow \mu_{\mathbb{C}}^{-1}(\iota^*\zeta_{\mathbb{C}})_{\iota^*\zeta_1}, \quad \sigma^{-1}(\zeta) \hookrightarrow \sigma_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})_{\zeta_1}$$

induce

$$\mu^{-1}(\iota^*\zeta)/H_\rho \to \mu_{\mathbb{C}}^{-1}(\iota^*\zeta_{\mathbb{C}})_{\iota^*\zeta_1}/H_\rho^{\mathbb{C}}, \quad \sigma^{-1}(\zeta)/H \to \sigma_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})_{\zeta_1}/H^{\mathbb{C}},$$

which are isomorphisms of complex analytic spaces by [8]. Here, to regard $\mu_{\mathbb{C}}^{-1}(\iota^*\zeta_{\mathbb{C}})_{\iota^*\zeta_1}/H_\rho^{\mathbb{C}}$ and $\sigma_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})_{\zeta_1}/H^{\mathbb{C}}$ as complex analytic spaces, we consider the sets of "semistable points" $\mu_{\mathbb{C}}^{-1}(\iota^*\zeta_{\mathbb{C}})_{\iota^*\zeta_1}^{ss}$ and $\sigma_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})_{\zeta_1}^{ss}$ in Section 4.3.

Thus the proof of Theorem 1.1 is reduced to construct an isomorphism between $\mu_{\mathbb{C}}^{-1}(\iota^*\zeta_{\mathbb{C}})_{\iota^*\zeta_1}/H_\rho^{\mathbb{C}}$ and $\sigma_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})_{\zeta_1}/H^{\mathbb{C}}$. First of all, we define an $H_\rho^{\mathbb{C}}$ equivariant holomorphic map $\hat{\psi} : \mu_{\mathbb{C}}^{-1}(\iota^*\zeta_{\mathbb{C}}) \to \sigma_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})$ so that a induced map $\psi : \mu_{\mathbb{C}}^{-1}(\iota^*\zeta_{\mathbb{C}})/H_\rho^{\mathbb{C}} \to \sigma_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})/H^{\mathbb{C}}$ is bijective.

Then it suffices to see that ψ gives a one-to-one correspondence between $\mu_{\mathbb{C}}^{-1}(\iota^*\zeta_{\mathbb{C}})_{\iota^*\zeta_1}/H_\rho^{\mathbb{C}}$ and $\sigma_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})_{\zeta_1}/H^{\mathbb{C}}$. To show it, we describe some equivalent conditions for x and $\hat{\psi}(x)$ to be $x \in \mu_{\mathbb{C}}^{-1}(\iota^*\zeta_{\mathbb{C}})_{\iota^*\zeta_1}$ and $\hat{\psi}(x) \in \sigma_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})_{\zeta_1}$ in Section 3, using some convex functions on $G\backslash G^{\mathbb{C}}$ We also need the description of the Kähler potential of N_G, which is discussed in Section 2.

This paper is organized as follows. We review the construction of hyper-Kähler structures on $N_G = T^*G^{\mathbb{C}}$ along [12], and describe hyper-Kähler moment map by [2] in Section 2. Moreover we describe the Kähler potentials of the hyper-Kähler metrics using the method of [9] and [5].

In Section 3, to obtain other description of $\mu_{\mathbb{C}}^{-1}(\iota^*\zeta_{\mathbb{C}})_{\iota^*\zeta_1}$ and $\sigma_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})_{\zeta_1}$, we study the relation between a Kähler moment map and some geodesically convex functions on Riemannian symmetric spaces.

In Section 4, we prove Theorem 1.1, by using the description of Kähler potentials obtained in Section 2 and the methods in Section 3.

In Section 5 we apply Theorem 1.1 to the Hilbert schemes of k points on $\mathbb{C}^2$ and show Theorem 1.2. Moreover, we see that Theorem 1.1 can be applied to quiver varieties and toric hyper-Kähler varieties.

2 Hyper-Kähler structures on $T^*G^{\mathbb{C}}$

2.1 Riemannian description

Here we review briefly the hyper-Kähler quotient construction of N_G along [12], and describe hyper-Kähler moment map ν along [2][3].

Let G be a compact connected Lie group, and $\|\cdot\|$ is a norm on $\mathbf{g}$ induced by an Ad_G-invariant inner product. Consider the following equations

$$\frac{dT_i}{ds} + [T_0, T_i] + [T_j, T_k] = 0 \quad \text{for } (i,j,k) = (1,2,3), (2,3,1), (3,1,2), \tag{2}$$

for $T := (T_0, T_1, T_2, T_3) \in C^1([0,1], \mathbf{g}) \otimes \mathbb{H}$. Put

$$\mathcal{N}_G := \{T \in C^1([0,1], \mathbf{g}) \otimes \mathbb{H};\ \text{equations (2) holds}\}.$$

Then a gauge group $\mathcal{G} := C^2([0,1], G)$ acts on $\mathcal{N}_G$ by

$$g \cdot T := (\mathrm{Ad}_g T_0 + g \frac{d}{ds} g^{-1}, \mathrm{Ad}_g T_1, \mathrm{Ad}_g T_2, \mathrm{Ad}_g T_3),$$

and we obtain

$$N_G := \mathcal{N}_G / \mathcal{G}_0,$$

where $\mathcal{G}_0 := \{g \in \mathcal{G};\ g(0) = g(1) = 1\}$. It is shown in [12] that N_G becomes a C^∞ manifold of dimesion $4 \dim G$, and the standard hyper-Kähler structure on $C^1([0,1], \mathbf{g}) \otimes \mathbb{H}$ induces a hyper-Kähler structure $g_G, I_{G,1}, I_{G,2}, I_{G,3}$ on N_G. Here, g_G is induced from the L^2-inner product on $C^1([0,1], \mathbf{g}) \otimes \mathbb{H}$ using Ad_G-invariant inner product on $\mathbf{g}$.

Now we have a Lie group isomorphism $\mathcal{G}/\mathcal{G}_0 = G \times G$ defined by $g \mapsto (g(0), g(1))$. Since $\mathcal{G}$ acts on $\mathcal{N}_G$, there exists a $G \times G$-action on N_G preserving the hyper-Kähler structure.

Theorem 2.1 ([2][3]). *The hyper-Kähler moment map $\nu = (\nu^0, \nu^1) : N_G \to \mathrm{Im}\mathbb{H} \otimes (\mathbf{g}^* \oplus \mathbf{g}^*)$ with respect to the action of $G \times G$ on N_G is given by*

$$\nu^0([T]) = (T_1(0), T_2(0), T_3(0)), \quad \nu^1([T]) = -(T_1(1), T_2(1), T_3(1)),$$

under the identification $\mathbf{g}^* \cong \mathbf{g}$ using Ad_G-invariant inner product. Here we denote by $[T] \in N_G$ the equivalence class represented by $T \in \mathcal{N}_G$.

2.2 Holomorphic description

In this subsection, we review that $(N_G, I_{G,1})$ is identified with a holomorphic cotangent bundle $T^*G^{\mathbb{C}} \cong G^{\mathbb{C}} \times \mathbf{g}^{\mathbb{C}}$ as complex manifolds along [5][12].

For each $T \in \mathcal{N}_G$, there exists a solution $u : [0,1] \to G^{\mathbb{C}}$ for an ordinary differential equation

$$\frac{du}{ds}u^{-1} = -(T_0 + \sqrt{-1}T_1),$$

then $T_2 + \sqrt{-1}T_3 = \mathrm{Ad}_{u(s)u(0)^{-1}}\eta$ for some $\eta \in \mathbf{g}^{\mathbb{C}}$ [12]. Then a holomorphic map $\Phi : (N_G, I_{G,1}) \to G^{\mathbb{C}} \times \mathbf{g}^{\mathbb{C}}$ is obtained by $[T] \mapsto (u(0)u(1)^{-1}, \eta)$.

Theorem 2.2 ([5][12]). *The map Φ is biholomorphic and preserves holomorphic symplectic structures.*

The moment map ν is decomposed into $\nu = (\nu_{\mathbb{R}} = \nu_1, \nu_{\mathbb{C}} = \nu_2 + \sqrt{-1}\nu_3)$ along the decomposition $\mathrm{Im}\mathbb{H} \cong \mathbb{R} \oplus \mathbb{C}$. Then

$$\nu_{\mathbb{C}}(Q, \eta) = (\eta, -\mathrm{Ad}_{Q^{-1}}(\eta))$$

under the identification $N_G = G^{\mathbb{C}} \times \mathbf{g}^{\mathbb{C}}$. $\nu_{\mathbb{C}}$ is a holomorphic moment map with respect to $G^{\mathbb{C}} \times G^{\mathbb{C}}$ action on $T^*G^{\mathbb{C}}$. This action is given by

$$(g_0, g_1)(Q, \eta) = (g_0 Q g_1^{-1}, \mathrm{Ad}_{g_0}\eta) \qquad (3)$$

for $(g_0, g_1) \in G^{\mathbb{C}} \times G^{\mathbb{C}}$ and $(Q, \eta) \in G^{\mathbb{C}} \times \mathbf{g}^{\mathbb{C}}$.

2.3 Kähler potentials

Next we describe the Kähler potential of the Kähler manifold $(N_G, g_G, I_{G,1})$. We apply the following results for N_G.

Lemma 2.3 ([9]). *Let (M, g, I_1, I_2, I_3) be a hyper-Kähler manifold with isometric S^1 action generated by a Killing field X, which satisfies*

$$L_X\omega_1 = \omega_2, \quad L_X\omega_2 = -\omega_1, \quad L_X\omega_3 = 0,$$

where ω_i is the Kähler form of (M, g, I_i). If μ is the moment map with respect to the S^1-action on (M, ω_3), then $\omega_1 = 2\sqrt{-1}\partial_1\overline{\partial}_1\mu$, $\omega_2 = 2\sqrt{-1}\partial_2\overline{\partial}_2\mu$.

We apply this lemma as follows. Let $\omega_{G,i} := g_G(I_{G,i}\cdot, \cdot)$, and define $e^{i\theta} \cdot [T] := [T_0, \cos\theta\, T_1 + \sin\theta\, T_2, -\sin\theta\, T_1 + \cos\theta\, T_2, T_3]$ for $[T] \in N_G$, which is an S^1-action on N_G preserving $\omega_{G,3}$ and satisfies the assumption of Lemma

(2.3). Then the moment map is given by $\|T_1\|_{L^2}^2 + \|T_2\|_{L^2}^2 = \int_0^1 (\|T_1(s)\|^2 + \|T_2(s)\|^2) ds$.

Moreover, we have another S^1-action defined by $e^{i\theta}\cdot[T] := [T_0, -\sin\theta\, T_3 + \cos\theta\, T_1, T_2, \cos\theta\, T_3 + \sin\theta\, T_1]$, which preserves $\omega_{G,2}$. In this case the moment map is given by $\|T_1\|_{L^2}^2 + \|T_3\|_{L^2}^2$. Thus we obtain the followings from Lemma 2.3

Proposition 2.4. *Put $\mathcal{E} : N_G \to \mathbb{R}$ to be*

$$\mathcal{E}([T]) := \|T_1\|_{L^2}^2 + \frac{1}{2}(\|T_2\|_{L^2}^2 + \|T_3\|_{L^2}^2).$$

Then $\omega_{G,1} = 2\sqrt{-1}\partial_1\overline{\partial}_1\mathcal{E}$.

Next we describe the Kähler potential $\mathcal{E}$ as a function on $T^*G^{\mathbb{C}} = G^{\mathbb{C}} \times \mathbf{g}^{\mathbb{C}}$. The Ad_G-invariant inner product on $\mathbf{g}$ induces a homogeneous Riemannian metric on $G\backslash G^{\mathbb{C}}$. Define an antiholomorphic involution of $G^{\mathbb{C}}$ by $(ge^{\sqrt{-1}\xi})^* := e^{\sqrt{-1}\xi}g^{-1}$ for $g \in G$ and $\xi \in \mathbf{g}$, by using the polar decomposition $G^{\mathbb{C}} \cong G\cdot\exp(\sqrt{-1}\mathbf{g})$. Then $G\backslash G^{\mathbb{C}}$ is identified with $\exp(\sqrt{-1}\mathbf{g})$ by $G\cdot g \mapsto g^*g$ for $g \in G^{\mathbb{C}}$. This metric on $G\backslash G^{\mathbb{C}}$ is naturally extended to a hermitian metric on $T_h(G\backslash G^{\mathbb{C}}) \otimes \mathbb{C}$, which is also denoted by $\|\cdot\|_h$.

Proposition 2.5. *For each $(Q,\eta) \in G^{\mathbb{C}} \times \mathbf{g}^{\mathbb{C}} = N_G$,*

$$\mathcal{E}(Q,\eta) = \frac{1}{2}\min_{h\in P(a^*a, b^*b)} \int_0^1 \left(\left\| \frac{dh}{ds} \right\|_h^2 + \|h\mathrm{Ad}_a^{-1}(\eta)\|_h^2 \right) ds,$$

where $ab^{-1} = Q$ and $P(h_0, h_1) := \{h \in C^\infty([0,1], G\backslash G^{\mathbb{C}});\ h(0) = h_0, h(1) = h_1\}$.

Proof. The essential part of the proof is obtained in [5], and we explain the outline. Define $\mathcal{L} : C^1([0,1],\mathbf{g}) \otimes \mathbb{H} \to \mathbb{R}$ by $\mathcal{L}(T) := \|T_1\|_{L^2}^2 + \frac{1}{2}(\|T_2\|_{L^2}^2 + \|T_3\|_{L^2}^2)$. Fix $T \in \mathcal{N}_G$, then $\mathcal{L}|_{\mathcal{G}_0^{\mathbb{C}}\cdot T}$ attains its minimum value at T by Lemma 2.3 of [5], hence

$$\mathcal{E}([T]) = \min_{g\in\mathcal{G}_0^{\mathbb{C}}} \mathcal{L}(g\cdot T).$$

Here $\mathcal{G}_0^{\mathbb{C}}$ is the complexified gauge group defined by

$$\mathcal{G}_0^{\mathbb{C}} := \{g \in \mathcal{G}^{\mathbb{C}} = C^2([0,1], G^{\mathbb{C}});\ g(0) = g(1) = 1\},$$

and a $\mathcal{G}_0^{\mathbb{C}}$ action on $C^1([0,1],\mathbf{g}) \otimes \mathbb{H}$ is defined by $g\cdot(T_0 + \sqrt{-1}T_1, T_2 + \sqrt{-1}T_3) := (\mathrm{Ad}_g(T_0 + \sqrt{-1}T_1) + g\frac{d}{ds}g^{-1}, \mathrm{Ad}_g(T_2 + \sqrt{-1}T_3))$. Take $u : [0,1] \to G^{\mathbb{C}}$ as in Section 2.2, and let $(Q,\eta) \in G^{\mathbb{C}} \times \mathbf{g}^{\mathbb{C}}$ be corresponding to $[T] \in N_G$

under the identification given in Section 2.2. Then we have $Q = u(0)u(1)^{-1}$, $T_2(s) + \sqrt{-1}T_3(s) = \mathrm{Ad}_{u(s)u(0)^{-1}}\eta$. For $g \in \mathcal{G}_0^{\mathbb{C}}$,

$$g \cdot T = \Big(g(s)u(s)\frac{d}{ds}(gu)^{-1}(s),\ \mathrm{Ad}_{g(s)u(s)}\mathrm{Ad}_{u(0)^{-1}}\eta\Big).$$

Now we extend the Ad_G-invariant inner product on $\mathbf{g}$ to the $\mathbb{C}$ bilinear form on $\mathbf{g}^{\mathbb{C}}$. From the calculation in the proof of Lemma 2.3 of [5], we obtain

$$\mathcal{L}(g \cdot T) = \int_0^1 (\frac{1}{4}\|h'\|_h^2 + \frac{1}{2}\|h\mathrm{Ad}_{u(0)^{-1}}(\eta)\|_h^2)ds,$$

where $h = (gu)^*(gu)$. Thus we obtain

$$\mathcal{E}([T]) = \min_{h \in P(h_0,h_1)} \int_0^1 (\frac{1}{4}\|h'\|_h^2 + \frac{1}{2}\|h\mathrm{Ad}_{u(0)^{-1}}(\eta)\|_h^2)ds,$$

where $h_0 = u(0)^*u(0)$, $h_1 = u(1)^*u(1)$ and $u(0)u(1)^{-1} = Q$, hence we have the assertion. $\square$

3 Kähler manifolds with Hamiltonian actions

Let $L^{\mathbb{C}}$ be a complexification of connected compact Lie group L, and $L^{\mathbb{C}}$ acts on a complex manifold (X, I) holomorphically. Let ω be a Kähler form on X, and L acts on (X, I, ω) isometrically. Suppose that there is a symplectic moment map $m : X \to \mathbf{l}^*$ with respect to L action, where $\mathbf{l}$ is the Lie algebra of L.

Fix an $\mathrm{Ad_L}$-invariant inner product on $\mathbf{l}$, then the Riemannian metric on the homogeneous space $L\backslash L^{\mathbb{C}}$ is induced. We denote by $Lg \in L\backslash L^{\mathbb{C}}$ the equivalence class represented by $g \in L^{\mathbb{C}}$, and put

$$\xi_g := \frac{d}{dt}\Big|_{t=0} Le^{\sqrt{-1}t\xi}g \in T_{Lg}(L\backslash L^{\mathbb{C}})$$

for each $\xi \in \mathbf{l}$, where $\{Le^{\sqrt{-1}t\xi}g\}_{t\in\mathbb{R}}$ is a geodesic through Lg.

Now define a 1-form $\alpha_{x,\zeta} \in \Omega^1(L\backslash L^{\mathbb{C}})$ by $(\alpha_{x,\zeta})_{Lg}(\xi_g) := \langle m(gx) - \zeta, \xi\rangle$ for $\zeta \in Z_L$, which is easily checked to be closed. Since $L\backslash L^{\mathbb{C}} \cong \mathbf{l}$ is simply-connected, $\alpha_{x,\zeta}$ is d-exact and there is a unique primitive function up to constant. Accordingly, there is a unique function $\Phi_{x,\zeta} : L\backslash L^{\mathbb{C}} \to \mathbb{R}$ satisfying $d\Phi_{x,\zeta} = \alpha_{x,\zeta}$ and $\Phi_{x,\zeta}(L \cdot 1) = 0$. Here, the latter equality is a normalization for removing the ambiguity, and it is not essential. Then it is easy to check that $\Phi_{x,\zeta}$ is a geodesically convex function on the Riemannian symmetric space $L\backslash L^{\mathbb{C}}$.

From the argument in [10][16], the naturally induced map

$$m^{-1}(\zeta)/L \to X_\zeta/L^{\mathbb{C}}$$

becomes a homeomorphism, where

$$X_\zeta := \{x \in X;\ \Phi_{x,\zeta} \text{ has a critical point}\}.$$

In this subsection we show some equivalent conditions for the existence of the critical point of $\Phi_{x,\zeta}$.

For $g \in L^{\mathbb{C}}$, we denote by R_g the isometry on $L\backslash L^{\mathbb{C}}$ given by the right action of $L^{\mathbb{C}}$ on $L\backslash L^{\mathbb{C}}$. Then we can check that $R_g^*\alpha_{x,\zeta} = \alpha_{gx,\zeta}$, and $R_g^*\Phi_{x,\zeta} - \Phi_{gx,\zeta}$ becomes a constant function, hence X_ζ is $L^{\mathbb{C}}$-closed.

Let

$$\mathrm{Stab}(x)^{\mathbb{C}} := \{g \in L^{\mathbb{C}};\ gx = x\},$$

and $\mathrm{stab}(x)^{\mathbb{C}}$ be the Lie algebra of $\mathrm{Stab}(x)^{\mathbb{C}}$. We put

$$\begin{aligned} \mathrm{Stab}(x) &:= \{g \in L;\ gx = x\} = \mathrm{Stab}(x)^{\mathbb{C}} \cap L, \\ \mathrm{stab}(x) &:= \mathrm{Lie}(\mathrm{Stab}(x)) = \mathrm{stab}(x)^{\mathbb{C}} \cap \mathbf{l}. \end{aligned}$$

Note that $\mathrm{Stab}(x)^{\mathbb{C}}$ contains the complexification of $\mathrm{Stab}(x)$ as a subgroup, though it is not necessary to be equal. Let $\pi_{\mathrm{Im}} : \mathbf{l}^{\mathbb{C}} \to \mathbf{l}$ be defined by $\pi_{\mathrm{Im}}(a + \sqrt{-1}b) = b$ for $a, b \in \mathbf{l}$, and put $\widetilde{\mathrm{stab}}(x) := \pi_{\mathrm{Im}}(\mathrm{stab}(x)^{\mathbb{C}})$. Then there is the orthogonal decomposition $\mathbf{l} = \widetilde{\mathrm{stab}}(x) \oplus V_x$ with respect to Ad_{L}-invariant inner product on $\mathbf{l}$.

Lemma 3.1. *For each $g \in L^{\mathbb{C}}$, there are $\gamma \in \mathrm{Stab}(x)^{\mathbb{C}}$ and $\xi \in V_x$ such that $Lg\gamma = Le^{\sqrt{-1}\xi}$.*

Proof. Consider the smooth function $f : Lg \cdot \mathrm{Stab}(x)^{\mathbb{C}} \to \mathbb{R}$ defined by

$$f(Lg\gamma) := \mathrm{dist}_{L\backslash L^{\mathbb{C}}}(L \cdot 1, Lg\gamma)^2,$$

where $Lg \cdot \mathrm{Stab}(x)^{\mathbb{C}} \subset L\backslash L^{\mathbb{C}}$ is the $\mathrm{Stab}(x)^{\mathbb{C}}$-orbit through Lg. Now $\mathrm{Stab}(x)^{\mathbb{C}}$ is closed in $L^{\mathbb{C}}$, then $Lg \cdot \mathrm{Stab}(x)^{\mathbb{C}}$ is the closed orbit, hence f is proper. Since f is bounded from below, there is a minimum point $Lg\gamma_0 \in Lg \cdot \mathrm{Stab}(x)^{\mathbb{C}}$. By the polar decomposition $L^{\mathbb{C}} \cong L \times \mathbf{l}$, we can take $h \in L$ and $\xi \in \mathbf{l}$ such that $hg\gamma_0 = e^{\sqrt{-1}\xi}$. Under the identification $T_{Lg\gamma_0}(L\backslash L^{\mathbb{C}}) \cong \mathbf{l}$, the subspace $T_{Lg\gamma_0}(Lg \cdot \mathrm{Stab}(x)^{\mathbb{C}})$ is identified with $\widetilde{\mathrm{stab}}(hg\gamma_0 x) = \widetilde{\mathrm{stab}}(hgx)$. Since the derivative of f at $Lg\gamma_0$ vanishes, we have $\xi \in V_{hg\gamma_0 x}$.

Take $\hat{b} \in \widetilde{\mathrm{stab}}(x)$ arbitrarily, and fix $\hat{a} \in \mathbf{l}$ such that $\hat{a}+\sqrt{-1}\hat{b} \in \mathrm{stab}(x)^{\mathbb{C}}$. Since $\mathrm{stab}(hg\gamma_0 x)^{\mathbb{C}} = \mathrm{Ad}_{hg\gamma_0}(\mathrm{stab}(x)^{\mathbb{C}})$, there is $a + \sqrt{-1}b \in \mathrm{stab}(hg\gamma_0 x)^{\mathbb{C}}$ and $a + \sqrt{-1}b = \mathrm{Ad}_{hg\gamma_0}(\hat{a} + \sqrt{-1}\hat{b})$. Since $b \in \widetilde{\mathrm{stab}}(hg\gamma_0 x)$, we have

$$\begin{aligned} 0 = 2\sqrt{-1}\langle \xi, b\rangle &= \langle \xi, \mathrm{Ad}_{hg\gamma_0}(\hat{a} + \sqrt{-1}\hat{b}) + (\mathrm{Ad}_{hg\gamma_0}(\hat{a} + \sqrt{-1}\hat{b}))^*\rangle \\ &= \langle \mathrm{Ad}_{(hg\gamma_0)^{-1}}\xi, \hat{a} + \sqrt{-1}\hat{b}\rangle + \langle \mathrm{Ad}_{(hg\gamma_0)^*}\xi, (\hat{a} + \sqrt{-1}\hat{b})^*\rangle \\ &= \langle \mathrm{Ad}_{e^{-\sqrt{-1}\xi}}\xi, \hat{a} + \sqrt{-1}\hat{b}\rangle + \langle \mathrm{Ad}_{e^{\sqrt{-1}\xi}}\xi, (\hat{a} + \sqrt{-1}\hat{b})^*\rangle \\ &= \langle \xi, \hat{a} + \sqrt{-1}\hat{b}\rangle + \langle \xi, -\hat{a} + \sqrt{-1}\hat{b}\rangle \\ &= 2\sqrt{-1}\langle \xi, \hat{b}\rangle. \end{aligned}$$

Thus we obtain $\xi \in V_x$. $\square$

Proposition 3.2. *$\Phi_{x,\zeta}$ has a critical point if and only if all of the following conditions are satisfied for some $g \in L^{\mathbb{C}}$; (i) $\Phi_{gx,\zeta}$ is $\mathrm{Stab}(gx)^{\mathbb{C}}$ invariant, (ii) $\lim_{t\to\infty} \Phi_{gx,\zeta}(Le^{\sqrt{-1}t\xi}) = \infty$ for any $\xi \in \mathbf{l} - \widetilde{\mathrm{stab}}(gx)$.*

Proof. Let $\Phi_{x,\zeta}$ has a critical point $Lg \in L\backslash L^{\mathbb{C}}$ for $g \in L^{\mathbb{C}}$. Since $\Phi_{gx,\zeta} - R_g^*\Phi_{x,\zeta}$ is a constant function, we may suppose $g = 1$ by the homogeneity. Then $\frac{d}{dt}|_{t=0}\Phi_{x,\zeta}(Le^{\sqrt{-1}t\xi}) = 0$ for all $\xi \in \mathbf{l}$. If $\xi \notin \widetilde{\mathrm{stab}}(x)$, especially $\xi \notin \mathrm{stab}(x)$,

$$\frac{d^2}{dt^2}\Big|_{t=0}\Phi_{x,\zeta}(Le^{\sqrt{-1}t\xi}) = \|\xi_x^*\|_\omega^2 > 0$$

and there exists sufficiently small $\delta > 0$ and $\frac{d}{dt}\Phi_{x,\zeta}(Le^{\sqrt{-1}t\xi}) \geq \delta$ for all $t \geq 1$. Thus we obtain $\lim_{t\to\infty} \Phi_{x,\zeta}(Le^{\sqrt{-1}t\xi}) = \infty$. For any $\xi \in \mathbf{l}$, we have $\frac{d}{dt}\Phi_{x,\zeta}(Le^{\sqrt{-1}t\xi}) \geq 0$ for any $t > 0$ and obtain $\Phi_{x,\zeta}(Le^{\sqrt{-1}\xi}) \geq \Phi_{x,\zeta}(L \cdot 1)$, hence $\Phi_{x,\zeta}(L\cdot 1)$ is the minimum value of $\Phi_{x,\zeta}$, especially $\Phi_{x,\zeta}$ is bounded from below. Next we show that $\Phi_{x,\zeta}$ is $\mathrm{Stab}(x)^{\mathbb{C}}$ invariant. For any $\gamma \in \mathrm{Stab}(x)^{\mathbb{C}}$, $R_\gamma^*\Phi_{x,\zeta} - \Phi_{x,\zeta}$ is a constant function. If we put $R_\gamma^*\Phi_{x,\zeta} - \Phi_{x,\zeta} = c_\gamma \in \mathbb{R}$, then we have $\Phi_{x,\zeta}(L\gamma) = \Phi_{x,\zeta}(L \cdot 1) + c_\gamma$ and $\Phi_{x,\zeta}(L\gamma^{-1}) = \Phi_{x,\zeta}(L \cdot 1) - c_\gamma$. Since $\Phi_{x,\zeta}(L \cdot 1)$ is a minimum value, then c_γ should be zero, which implies $\Phi_{x,\zeta}$ is $\mathrm{Stab}(x)^{\mathbb{C}}$ invariant.

Conversely, assume that the conditions (i)-(ii) hold. It suffices to show that $\Phi_{gx,\zeta}$ has a minimum point in $L\backslash L^{\mathbb{C}}$. To see it, it suffices to see that $\Phi_{gx,\zeta}|_{\exp(V_{gx})}$ has a minimum point by applying Lemma 3.1, where

$$\exp(V_{gx}) \ := \ \{L \cdot e^{\sqrt{-1}\xi} \in L\backslash L^{\mathbb{C}};\ \xi \in V_{gx}\}$$

and $\mathbf{l} = \widetilde{\mathrm{stab}}(gx) \oplus V_{gx}$ is the orthogonal decomposition with respect to the Ad_L-invariant inner product. Define a smooth function $F : S(V_{gx}) \times \mathbb{R} \to \mathbb{R}$

by

$$F(\xi, t) := \frac{d}{dt}\Phi_{gx,\zeta}(e^{\sqrt{-1}t\xi}),$$

where $S(V_{gx}) := \{\xi \in V_{gx}; \|\xi\| = 1\}$. Now, recall that $t \mapsto \Phi_{gx,\zeta}(e^{\sqrt{-1}t\xi})$ is convex and we have assumed that $\lim_{|t|\to\infty} \Phi_{gx,\zeta}(e^{\sqrt{-1}t\xi}) = \infty$ for all $\xi \in S(V_{gx})$. It implies that there exists a unique $\hat{t}(\xi) \in \mathbb{R}$ for each $\xi \in S(V_{gx})$ such that $F(\xi, \hat{t}(\xi)) = 0$. Since $\frac{\partial}{\partial t}F(\xi, t) = \|\xi^*_x\|^2_\omega > 0$, $\hat{t} : S(V_{gx}) \to \mathbb{R}$ is smooth by Implicit Function Theorem. In particular, $\xi \mapsto \Phi_{gx,\zeta}(e^{\sqrt{-1}\hat{t}(\xi)\xi})$ becomes a smooth function on the compact manifold $S(V_{gx})$, hence it has a minimum point $\xi_{min} \in S(V_{gx})$. It is easy to see $\exp(\hat{t}(\xi_{min})\xi_{min}) \in \exp(V_{gx})$ is a minimum point of $\Phi_{gx,\zeta}|_{\exp(V_{gx})}$. □

Since each $\mathrm{Stab}(x)^{\mathbb{C}}$-orbit is closed in $L\backslash L^{\mathbb{C}}$, the distance on $L\backslash L^{\mathbb{C}}$ induces a structure of a metric space on $L\backslash L^{\mathbb{C}}/\mathrm{Stab}(x)^{\mathbb{C}}$. If $\Phi_{x,\zeta}$ is $\mathrm{Stab}(x)^{\mathbb{C}}$-invariant, then it induces a function $\bar{\Phi}_{x,\zeta} : L\backslash L^{\mathbb{C}}/\mathrm{Stab}(x)^{\mathbb{C}} \to \mathbb{R}$.

Proposition 3.3. *$\Phi_{x,\zeta}$ has a critical point if and only if $\Phi_{x,\zeta}$ is $\mathrm{Stab}(x)^{\mathbb{C}}$-invariant, and $\bar{\Phi}_{x,\zeta}$ is proper and bounded from below.*

Proof. Assume that $\Phi_{x,\zeta}$ has a critical point. Then the conditions (i)-(ii) in Proposition 3.2 are satisfied for some $g \in L^{\mathbb{C}}$, accordingly it suffices to show the properness of $\bar{\Phi}_{gx,\zeta}$. Define an equivalence relation in $\exp(V_{gx})$ by $Lg_1 \sim Lg_2$ if Lg_1 and Lg_2 lie on the same $\mathrm{Stab}(x)^{\mathbb{C}}$ orbit. Then the homeomorphism $\bar{\Phi}_{x,\zeta} : \exp(V_{gx})/\sim \to L\backslash L^{\mathbb{C}}/\mathrm{Stab}(x)^{\mathbb{C}}$ is naturally induced. Since $\Phi_{x,\zeta}|_{\exp(V_{gx})}$ is proper, $\bar{\Phi}_{gx,\zeta}$ is also proper.

Conversely, assume that $\Phi_{x,\zeta}$ is $\mathrm{Stab}(x)^{\mathbb{C}}$-invariant, and that $\bar{\Phi}_{x,\zeta}$ is proper and bounded from below. Then the minimizing sequence of $\bar{\Phi}_{x,\zeta}$ always converges, therefore $\Phi_{x,\zeta}$ has a minimum point. □

Proposition 3.4. *Assume that $(d\Phi_{x,\zeta})_{L\cdot 1} = 0$. Let a diffeomorphism $\Psi_L : L \times \mathbf{l} \to L^{\mathbb{C}}$ be defined by $\Psi_L(g, \xi) := ge^{\sqrt{-1}\xi}$. Then the restriction*

$$\Psi_L|_{\mathrm{Stab}(x)\times\mathrm{stab}(x)} : \mathrm{Stab}(x) \times \mathrm{stab}(x) \to \mathrm{Stab}(x)^{\mathbb{C}}$$

is a diffeomorphism.

Proof. Let $(d\Phi_{x,\zeta})_{L\cdot 1} = 0$. Take $\gamma \in \mathrm{Stab}(x)^{\mathbb{C}}$, and put $\gamma = ge^{\sqrt{-1}\xi}$ for some $g \in L$ and $\xi \in \mathbf{l}$. It suffices to show $g \in \mathrm{Stab}(x)$ and $\xi \in \mathrm{stab}(x)$. By the proof of Proposition 3.2, we have $\Phi_{x,\zeta}(L \cdot 1) = \Phi_{x,\zeta}(L \cdot \gamma) = \Phi_{x,\zeta}(L \cdot e^{\sqrt{-1}\xi})$. Since $\Phi_{x,\zeta}$ is geodesically convex and $\{L \cdot e^{\sqrt{-1}t\xi}\}_{t\in\mathbb{R}}$ is a geodesic, $\Phi_{x,\zeta}$ have to be constant on this geodesic. Thus we have

$$0 = \frac{d^2}{dt^2}\Big|_{t=0}\Phi_{x,\zeta}(e^{\sqrt{-1}t\xi}) = \|\xi^*_x\|^2,$$

which implies $\xi \in \mathrm{stab}(x)$. Then we obtain

$$g = \gamma e^{-\sqrt{-1}t\xi} \in \mathrm{Stab}(x)^{\mathbb{C}} \cap L = \mathrm{Stab}(x).$$

$\square$

Remark 3.1. It is shown by Corollary 2.15 in [17] that $\mathrm{stab}(x)^{\mathbb{C}}$ are reductive for all x such that $(d\Phi_{x,\zeta})_{L\cdot 1} = 0$.

Next we assume that there is an L-invariant function $\varphi \in C^\infty(L\backslash L^{\mathbb{C}})$ and satisfies $\omega = \sqrt{-1}\partial\bar{\partial}\varphi$. Then we have $\Phi_{x,\zeta}(L\cdot e^{\sqrt{-1}\xi}) = \varphi(e^{\sqrt{-1}\xi}\cdot x) - \langle\zeta,\xi\rangle + c$ for some constant c by the discussion of (2.6) in [8]. Here we may assume $c = 0$, since the existence of the critical point does not depend on the value of c.

Proposition 3.5. *Assume that* $\lim_{t\to\infty}\varphi(e^{\sqrt{-1}t\xi}x)/t = \infty$ *holds for all* $\xi \in \mathbf{l}$ *which satisfy* $\lim_{t\to\infty}\varphi(e^{\sqrt{-1}t\xi}x) = \infty$. *If* $\Phi_{x,\zeta}$ *has a critical point, then* $\Phi_{x,s\zeta}$ *also has a critical point for each* $s > 0$.

Proof. We may assume the conditions $(i)(ii)$ of Proposition 3.2 are satisfied for $\Phi_{x,\zeta}$. It suffices to show that $\Phi_{x,s\zeta}$ also satisfies $(i)(ii)$. Since $\Phi_{x,\zeta}$ is $\mathrm{Stab}(x)^{\mathbb{C}}$ invariant, we have

$$\begin{aligned}\Phi_{x,s\zeta}(Lg\gamma) &= s\Phi_{x,\zeta}(Lg\gamma) + (1-s)\varphi(g\gamma x)\\ &= s\Phi_{x,\zeta}(Lg) + (1-s)\varphi(gx) = \Phi_{x,s\zeta}(Lg)\end{aligned}$$

for all $Lg \in L\backslash L^{\mathbb{C}}$ and $\gamma \in \mathrm{Stab}(x)^{\mathbb{C}}$, thus $\Phi_{x,s\zeta}$ is $\mathrm{Stab}(x)^{\mathbb{C}}$ invariant. Next we take $\xi \in \mathbf{l} - \widetilde{\mathrm{stab}}(x)$ and consider the behavior of $\Phi_{x,s\zeta}(e^{\sqrt{-1}t\xi})$ for $t \to \infty$. Since $\lim_{t\to\infty}\Phi_{x,\zeta}(e^{\sqrt{-1}t\xi}) = \infty$, we have $\lim_{t\to\infty}\varphi(e^{\sqrt{-1}t\xi}) = \infty$ or $-\langle\zeta,\xi\rangle > 0$. If the latter case occurs, then we obtain $\lim_{t\to\infty}\Phi_{x,s\zeta}(e^{\sqrt{-1}t\xi}) = \infty$. Let the former case occur. From the assumption we have $\varphi(e^{\sqrt{-1}t\xi}x)/t \to \infty$ for $t \to \infty$, thus $\lim_{t\to\infty}\Phi_{x,s\zeta}(e^{\sqrt{-1}t\xi}) = \infty$ for all $s > 0$. $\square$

4 Main results

4.1 Correspondence of orbits

In this section we prove Theorem 1.1. Let (M, g, I_1, I_2, I_3), H, G, ρ and H_ρ be as in Section 1.2. First of all, we show the following lemma.

Lemma 4.1. *Let* $\bar{\rho}$ *be defined as (1), and assume it is surjective. Then a linear map*

$$\rho^*|_{\mathrm{Ann}.\Delta_{\mathbf{g}}} : \mathrm{Ann}.\Delta_{\mathbf{g}} \to \mathrm{Ann}.\mathbf{h}_\rho$$

is bijective, where

$$\begin{aligned}\mathrm{Ann}.\Delta_{\mathbf{g}} &:= \{\varphi \in \mathbf{g}^* \oplus \mathbf{g}^*;\ \varphi|_{\Delta_{\mathbf{g}}} = 0\},\\ \mathrm{Ann}.\mathbf{h}_\rho &:= \{\varphi \in \mathbf{h}^*;\ \varphi|_{\mathbf{h}_\rho} = 0\}.\end{aligned}$$

Proof. The assertion is obvious since $\rho^*|_{\mathrm{Ann}.\Delta_{\mathbf{g}}}$ is the adjoint map of

$$\bar{\rho}_* : \mathbf{h}/\mathbf{h}_\rho \to (\mathbf{g} \oplus \mathbf{g})/\Delta_{\mathbf{g}},$$

under the identification $\{\mathbf{h}/\mathbf{h}_\rho\}^* \cong \mathrm{Ann}.\mathbf{h}_\rho$ and $\{(\mathbf{g} \oplus \mathbf{g})/\Delta_{\mathbf{g}}\}^* \cong \mathrm{Ann}.\Delta_{\mathbf{g}}$. $\square$

Any $x \in \mu_{\mathbb{C}}^{-1}(\iota^*\zeta_{\mathbb{C}})$ satisfies $\hat{\mu}_{\mathbb{C}}(x) - \zeta_{\mathbb{C}} \in \ker \iota^* = \mathrm{Ann}.\mathbf{h}_\rho^{\mathbb{C}}$ by the definition. Consequently, there exists a unique $\eta(x) \in (\mathbf{g}^{\mathbb{C}})^*$ such that

$$\hat{\mu}_{\mathbb{C}}(x) - \zeta_{\mathbb{C}} = \rho^*(-\eta(x), \eta(x))$$

by Lemma 4.1, which implies $(x, 1, \eta(x)) \in \sigma_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})$. Here we identify $N_G = G^{\mathbb{C}} \times \mathbf{g}^{\mathbb{C}}$ and $(\mathbf{g}^{\mathbb{C}})^* = \mathbf{g}^{\mathbb{C}}$ by the Ad_G-invariant $\mathbb{C}$ bilinear form. Thus we obtain a map $\hat{\psi} : \mu_{\mathbb{C}}^{-1}(\iota^*\zeta_{\mathbb{C}}) \to \sigma_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})$ defined by $\hat{\psi}(x) := (x, 1, \eta(x))$, which induces a map $\psi : \mu_{\mathbb{C}}^{-1}(\iota^*\zeta_{\mathbb{C}})/H_\rho^{\mathbb{C}} \to \sigma_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})/H^{\mathbb{C}}$.

Proposition 4.2. *ψ is well-defined and a homeomorphism.*

Proof. First of all we check the well-definedness. Let $x \in \mu_{\mathbb{C}}^{-1}(\iota^*\zeta_{\mathbb{C}})$ and $h \in H_\rho^{\mathbb{C}}$. Now we may write $\rho(h) = (h_0, h_1) \in G^{\mathbb{C}} \times G^{\mathbb{C}}$, and suppose $h_0 = h_1$. Then $hx \in \mu_{\mathbb{C}}^{-1}(\iota^*\zeta_{\mathbb{C}})$ and $\hat{\psi}(hx) = (hx, 1, \eta(hx))$, where $\eta(hx) \in \mathbf{g}^{\mathbb{C}}$ is uniquely defined by $\hat{\mu}_{\mathbb{C}}(hx) - \zeta_{\mathbb{C}} = \rho^*(-\eta(hx), \eta(hx))$. Now we have

$$\begin{aligned}\hat{\mu}_{\mathbb{C}}(hx) - \zeta_{\mathbb{C}} &= \mathrm{Ad}^*_{h^{-1}}(\hat{\mu}_{\mathbb{C}}(x) - \zeta_{\mathbb{C}})\\ &= \mathrm{Ad}^*_{h^{-1}}\rho^*(-\eta(x), \eta(x))\\ &= \rho^*(-\mathrm{Ad}^*_{h_0^{-1}}\eta(x), \mathrm{Ad}^*_{h_0^{-1}}\eta(x)),\end{aligned}$$

where $\mathrm{Ad}^*_g \in GL((\mathbf{g}^{\mathbb{C}})^*)$ is defined by

$$\langle \mathrm{Ad}^*_g y, \xi\rangle := \langle y, \mathrm{Ad}_g \xi\rangle$$

for $y \in (\mathbf{g}^{\mathbb{C}})^*, \xi \in \mathbf{g}^{\mathbb{C}}, g \in G^{\mathbb{C}}$, hence $\eta(hx) = \mathrm{Ad}^*_{h_0^{-1}}\eta(x)$ holds by the uniqueness. Since $\mathrm{Ad}^*_{h_0^{-1}}$ corresponds to Ad_{h_0} under the identification $(\mathbf{g}^{\mathbb{C}})^* \cong \mathbf{g}^{\mathbb{C}}$, we obtain $\hat{\psi}(hx) = h\hat{\psi}(x)$.

Next we show the injectivity. Take $x, x' \in \mu_{\mathbb{C}}^{-1}(\iota^*\zeta_{\mathbb{C}})$ to be $\hat{\psi}(x) = h\hat{\psi}(x')$ for some $h \in H^{\mathbb{C}}$. Then $(x, 1, \eta(x)) = (hx', h_0h_1^{-1}, \mathrm{Ad}_{h_0}\eta(x'))$, accordingly we obtain $h_0h_1^{-1} = 1$ which implies $h \in H_\rho^{\mathbb{C}}$.

The surjectivity is shown by constructing the inverse map of ψ as follows. Take $(x, Q, \eta) \in \sigma_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})$ arbitrarily. From Section 2.2, we have

$$\begin{aligned} \sigma_{\mathbb{C}}(x, Q, \eta) &= \hat{\mu}_{\mathbb{C}}(x) + \rho^*(\nu(Q, \eta)) \\ &= \hat{\mu}_{\mathbb{C}}(x) + \rho^*(\eta, -\mathrm{Ad}_{Q^{-1}}\eta) = \zeta_{\mathbb{C}} \end{aligned} \tag{4}$$

From the surjectivity of the map (1), there exist some $h \in H^{\mathbb{C}}$ such that $h_0h_1^{-1} = Q$. Then $\rho(h)^{-1} = (h_0^{-1}, h_0^{-1}Q) \in H^{\mathbb{C}}$, and we have

$$\begin{aligned} \rho^*(\eta, -\mathrm{Ad}_{Q^{-1}}\eta) &= \rho^*\mathrm{Ad}_{\rho(h)}(\mathrm{Ad}_{h_0^{-1}}\eta, -\mathrm{Ad}_{h_0^{-1}}\eta) \\ &= \mathrm{Ad}_h\rho^*(\mathrm{Ad}_{h_0^{-1}}\eta, -\mathrm{Ad}_{h_0^{-1}}\eta). \end{aligned} \tag{5}$$

By combining (4)(5), we obtain

$$\hat{\mu}_{\mathbb{C}}(h^{-1}x) + \rho^*(\mathrm{Ad}_{h_0^{-1}}\eta, -\mathrm{Ad}_{h_0^{-1}}\eta) = \zeta_{\mathbb{C}},$$

which means $h^{-1}x \in \mu_{\mathbb{C}}^{-1}(\iota^*\zeta_{\mathbb{C}})$ and

$$\hat{\psi}(h^{-1}x) = (h^{-1}x, 1, \mathrm{Ad}_{h_0^{-1}}\eta) = h^{-1}(x, Q, \eta).$$

Thus we have the surjectivity of ψ. Here, we can take h depending on Q continuously in local, therefore the inverse of ψ becomes continuous. □

We can give group isomorphisms between the stabilizers as follows. Let

$$\begin{aligned} \mathrm{Stab}(x)^{\mathbb{C}} &:= \{g \in H_\rho^{\mathbb{C}};\ gx = x\}, \\ \mathrm{Stab}(x, Q, \eta)^{\mathbb{C}} &:= \{g \in H^{\mathbb{C}};\ g(x, Q, \eta) = (x, Q, \eta)\}. \end{aligned}$$

Then it is easy to check that the inclusion $\mathrm{Stab}(x)^{\mathbb{C}} \hookrightarrow \mathrm{Stab}(\hat{\psi}(x))^{\mathbb{C}}$ is surjective, hence we obtain a Lie group isomorphism

$$\mathrm{Stab}(x)^{\mathbb{C}} \cong \mathrm{Stab}(\hat{\psi}(x))^{\mathbb{C}}. \tag{6}$$

4.2 Correspondence of stability

Put

$$\begin{aligned} \mu_{\mathbb{C}}^{-1}(\iota^*\zeta_{\mathbb{C}})_{\iota^*\zeta_1} &= \{x \in \mu_{\mathbb{C}}^{-1}(\iota^*\zeta_{\mathbb{C}});\ \Phi_{x,\iota^*\zeta_1} \text{ has a critical point}\}, \\ \sigma_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})_{\zeta_1} &= \{(x, Q, \eta) \in \sigma_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}});\ \Phi_{(x,Q,\eta),\zeta_1} \text{ has a critical point}\}. \end{aligned}$$

In this subsection we prove that ψ is a bijection from $\mu_{\mathbb{C}}^{-1}(\iota^*\zeta_{\mathbb{C}})_{\iota^*\zeta_1}/H_\rho^{\mathbb{C}}$ to $\sigma_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})_{\zeta_1}/H^{\mathbb{C}}$ by using the results in Section 3.

Along Section 3, we define geodesically convex functions

$$\Phi_{x,\zeta_1} : H\backslash H^{\mathbb{C}} \to \mathbb{R}, \quad \Phi_{x,\iota^*\zeta_1} : H_\rho\backslash H_\rho^{\mathbb{C}} \to \mathbb{R}, \quad \Phi_{(x,Q,\eta),\zeta_1} : H\backslash H^{\mathbb{C}} \to \mathbb{R},$$

for $x \in M$ and $(x, Q, \eta) \in M \times N_G$, corresponding to the moment maps $m = \hat{\mu}_1, \mu_1, \sigma_1$, respectively. Since H_ρ is a closed subgroup of H, $H_\rho\backslash H_\rho^{\mathbb{C}}$ is naturally embedded in $H\backslash H^{\mathbb{C}}$. Then we have $\Phi_{x,\iota^*\zeta_1}(H_\rho h) = \Phi_{x,\zeta_1}(Hh)$ for all $h \in H_\rho^{\mathbb{C}}$. Moreover we may write $\Phi_{(x,Q,\eta),\zeta_1}(H\hat{h}) = \Phi_{x,\zeta_1}(H\hat{h}) + \mathcal{E}(\hat{h}_0 Q \hat{h}_1^{-1}, \mathrm{Ad}_{\hat{h}_0}\eta)$ for all $\hat{h} \in H^{\mathbb{C}}$ from Proposition 2.4 and (2.6) in [8].

Proposition 4.3. *Let $x \in \mu_{\mathbb{C}}^{-1}(\iota^*\zeta_{\mathbb{C}})_{\iota^*\zeta_1}$. Then $\hat{\psi}(x) \in \sigma_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})_{\zeta_1}$.*

Proof. It suffices to show that $\Phi_{\hat{\psi}(x),\zeta_1}$ has a critical point if $\Phi_{x,\iota^*\zeta_1}$ has a critical point.

First of all, it is easy to check that $\Phi_{\hat{\psi}(x),\zeta_1}$ is $\mathrm{Stab}(\hat{\psi}(x))^{\mathbb{C}}$ invariant, since $\mathrm{Stab}(x)^{\mathbb{C}} = \mathrm{Stab}(\hat{\psi}(x))^{\mathbb{C}}$ and $\Phi_{x,\iota^*\zeta_1}$ is $\mathrm{Stab}(x)^{\mathbb{C}}$ invariant.

Next we take $\hat{\xi} \in \mathbf{h}$, put $\rho_*(\hat{\xi}) = (\hat{\xi}_0, \hat{\xi}_1)$ and consider the behavior of $\Phi_{\hat{\psi}(x),\zeta_1}(He^{\sqrt{-1}t\hat{\xi}})$ for $t \to \infty$. Since Φ_{x,ζ_1} is geodesically convex, there is a constant $c_{\hat{\xi}}\mathbb{R}$ and $\liminf_{t\to+\infty} \frac{d}{dt}\Phi_{x,\zeta_1}(He^{\sqrt{-1}t\hat{\xi}}) \geq c_{\hat{\xi}}$. Then we have an inequality $\Phi_{x,\zeta_1}(He^{\sqrt{-1}t\hat{\xi}}) \geq c_{\hat{\xi}}t - N_1$ for all $t \in \mathbb{R}$, for some sufficiently large N_1. If $\hat{\xi}_0 \neq \hat{\xi}_1$, then

$$\begin{aligned}
\Phi_{\hat{\psi}(x),\zeta_1}(He^{\sqrt{-1}t\hat{\xi}}) &\geq \mathcal{E}(e^{\sqrt{-1}t\hat{\xi}_0}e^{-\sqrt{-1}t\hat{\xi}_1}, \mathrm{Ad}_{e^{\sqrt{-1}t\hat{\xi}_0}}\eta(x)) + c_{\hat{\xi}}t - N_1 \\
&\geq \min_{h\in P(e^{2\sqrt{-1}t\hat{\xi}_0}, e^{-2\sqrt{-1}t\hat{\xi}_1})} \int_0^1 \frac{1}{4}\|h'\|_h^2 + c_{\hat{\xi}}t - N_1 \\
&\geq \mathrm{dist}_{G\backslash G^{\mathbb{C}}}(e^{2\sqrt{-1}t\hat{\xi}_0}, e^{2\sqrt{-1}t\hat{\xi}_1})^2 + c_{\hat{\xi}}t - N_1
\end{aligned}$$

Now $G\backslash G^{\mathbb{C}}$ is an Hadamard manifold, therefore the function

$$t \mapsto \mathrm{dist}_{G\backslash G^{\mathbb{C}}}(e^{2\sqrt{-1}t\hat{\xi}_0}, e^{2\sqrt{-1}t\hat{\xi}_1})$$

is convex. Since $\hat{\xi}_0 \neq \hat{\xi}_1$, there exists a positive constant $N_2 > 0$ and

$$\mathrm{dist}_{G\backslash G^{\mathbb{C}}}(e^{2\sqrt{-1}t\hat{\xi}_0}, e^{2\sqrt{-1}t\hat{\xi}_1})^2 \geq N_2 t^2$$

for $t \geq 1$, and we obtain $\Phi_{\hat{\psi}(x),\zeta_1}(He^{\sqrt{-1}t\hat{\xi}}) \to \infty$ for $t \to \infty$.

If $\hat{\xi}_0 = \hat{\xi}_1$, then $\hat{\xi} \in \mathbf{h}_\rho$. In this case we have

$$\Phi_{\hat{\psi}(x),\zeta_1}(He^{\sqrt{-1}t\hat{\xi}}) \geq \Phi_{x,\iota^*\zeta_1}(H_\rho e^{\sqrt{-1}t\hat{\xi}}) \to \infty$$

for $t \to \infty$, if we take $\hat{\xi} \notin \widetilde{\mathrm{stab}}(\hat{\psi}(x)) = \widetilde{\mathrm{stab}}(x)$, where the $\widetilde{\mathrm{stab}}$ is defined in the next section. Thus $\Phi_{\hat{\psi}(x),\zeta_1}$ has a critical value by Proposition 3.2. $\square$

Next we show the converse correspondence. From now on, we assume that there is an H-invariant global Kähler potential $\varphi : M \to \mathbb{R}$ of (M, I_1, ω_1), then we have

$$\begin{aligned} \Phi_{x,\iota^*\zeta_1}(H_\rho e^{\sqrt{-1}\xi}) &= \varphi(e^{\sqrt{-1}\xi}x) - \langle \iota^*\zeta_1, \xi\rangle + const.\ , \\ \Phi_{(x,Q,\eta),\zeta_1}(He^{\sqrt{-1}\hat{\xi}}) &= \varphi(e^{\sqrt{-1}\hat{\xi}}x) + \mathcal{E}(Q,\eta) - \langle \zeta_1, \hat{\xi}\rangle + const.\ , \end{aligned}$$

where $\xi \in \mathbf{h}_\rho$ and $\hat{\xi} \in \mathbf{h}$. Here we may assume the constant in the right hand sides of equalities are equal to 0.

Proposition 4.4. *Assume that there exists a smooth function $q : \mathbb{R} \to \mathbb{R}$ such that $\|\eta(x)\|^2 \le q(\varphi(x))$ and $q'(\varphi(x)) \ge 0$ for any $x \in M$. Suppose that if $\varphi(e^{\sqrt{-1}t\xi} \cdot x) \to \infty$ for $t \to \infty$ then $\lim_{t\to\infty} \varphi(e^{\sqrt{-1}t\xi} \cdot x)/t = \infty$ for any $\xi \in \mathbf{h}_\rho$. If $\hat{\psi}(x) \in \sigma_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})_{\zeta_1}$, then $x \in \mu_{\mathbb{C}}^{-1}(\iota^*\zeta_{\mathbb{C}})_{\iota^*\zeta_1}$.*

Proof. Assume that $\Phi_{\hat{\psi}(x),\zeta_1}$ has a critical point. From Proposition 3.3, $\Phi_{\hat{\psi}(x),\zeta_1}$ is $\mathrm{Stab}(\hat{\psi}(x))^{\mathbb{C}}$ invariant and the induced map

$$\bar{\Phi}_{\hat{\psi}(x),\zeta_1} : H\backslash H^{\mathbb{C}}/\mathrm{Stab}(\hat{\psi}(x))^{\mathbb{C}} \to \mathbb{R}$$

is proper, and bounded from below. Since $H_\rho\backslash H_\rho^{\mathbb{C}}/\mathrm{Stab}(x)^{\mathbb{C}}$ is a closed subset of $H\backslash H^{\mathbb{C}}/\mathrm{Stab}(\hat{\psi}(x))^{\mathbb{C}}$, $F := \bar{\Phi}_{\hat{\psi}(x),\zeta_1}|_{H_\rho\backslash H_\rho^{\mathbb{C}}/\mathrm{Stab}(x)^{\mathbb{C}}}$ is also proper and bounded below.

If $(Q,\eta) := (1, \eta(gx))$ for $g \in H_\rho^{\mathbb{C}}$, we have an upper estimate

$$\mathcal{E}(1, \eta(gx)) \le \|\eta(x)\|^2 \le q(\varphi(gx)),$$

by taking a path $h \in P(1,1)$ to be $h(s) = 1$. Hence if we take $\xi \in \mathbf{h}_\rho$, then

$$\Phi_{\hat{\psi}(x),\zeta_1}(He^{\sqrt{-1}\xi}) \le \Phi_{x,\iota^*\zeta_1}(H_\rho e^{\sqrt{-1}\xi}) + q(\varphi(e^{\sqrt{-1}\xi}x)) =: \hat{F}_+(H_\rho e^{\sqrt{-1}\xi}).$$

Now $\hat{F}_+$ induces a function $F_+ : H_\rho\backslash H_\rho^{\mathbb{C}}/\mathrm{Stab}(x)^{\mathbb{C}} \to \mathbb{R}$, which satisfies $F_+ \ge F$, therefore F_+ is also proper and bounded from below. Thus $\hat{F}_+$ has a minimum point $e^{\sqrt{-1}\xi} \in H_\rho\backslash H_\rho^{\mathbb{C}}$, and have

$$0 = (d\hat{F}_+)_{e^{\sqrt{-1}\xi}} = (1 + q'(\varphi(e^{\sqrt{-1}\xi}x)))\mu(e^{\sqrt{-1}\xi}x) - \iota^*\zeta_1.$$

Now we have shown that $\Phi_{x,s\cdot\iota^*\zeta_1}$ has a critical point if we put $s = 1 + q'(\varphi(e^{\sqrt{-1}\xi}x))$, hence $\Phi_{x,\iota^*\zeta_1}$ also has a critical point by Proposition 3.5. $\square$

Remark 4.1. The assumption of Proposition 4.4 is always satisfied if M is the quaternionic vector space $\mathbb{H}^N$ with Euclidean metric, and $H \subset Sp(N)$ acts on M linearly.

4.3 Proof of the main theorem

Proposition 4.5. $\mathrm{Stab}(x) \subset H_\rho$ *and* $\mathrm{Stab}(y) \subset H$ *are isomorphic as Lie groups for any* $x \in \mu^{-1}(\iota^*\zeta)$ *and* $y \in \sigma^{-1}(\zeta)$ *satisfying* $yH = \psi(xH_\rho)$.

Proof. The assertion follows directly from Proposition 3.4 and the isomorphism (6). □

Proof of Theorem 1.1. Define an open subsets $\mu_{\mathbb{C}}^{-1}(\iota^*\zeta_{\mathbb{C}})^{ss}_{\iota^*\zeta_1} \subset \mu_{\mathbb{C}}^{-1}(\iota^*\zeta_{\mathbb{C}})$ and $\sigma_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})^{ss}_{\zeta_1} \subset \sigma_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})$ by

$$\begin{aligned} \mu_{\mathbb{C}}^{-1}(\iota^*\zeta_{\mathbb{C}})^{ss}_{\iota^*\zeta_1} &:= \{x \in \mu_{\mathbb{C}}^{-1}(\iota^*\zeta_{\mathbb{C}});\ \overline{H_\rho^{\mathbb{C}} \cdot x} \cap \mu_1^{-1}(\iota^*\zeta_1) \neq \varnothing\}, \\ \sigma_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})^{ss}_{\zeta_1} &:= \{y \in \mu_{\mathbb{C}}^{-1}(\iota^*\zeta_{\mathbb{C}});\ \overline{H^{\mathbb{C}} \cdot y} \cap \sigma_1^{-1}(\zeta_1) \neq \varnothing\}. \end{aligned}$$

Then the naturally induced maps

$$\mu^{-1}(\iota^*\zeta)/H_\rho \to \mu_{\mathbb{C}}^{-1}(\iota^*\zeta_{\mathbb{C}})^{ss}_{\iota^*\zeta_1}//H_\rho^{\mathbb{C}}, \quad \sigma^{-1}(\zeta)/H \to \sigma_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})^{ss}_{\zeta_1}//H^{\mathbb{C}}$$

gives an isomorphisms as complex analytic spaces by the main theorem in [8], where $//$ is the categorical quotient. Moreover $\mu_{\mathbb{C}}^{-1}(\iota^*\zeta_{\mathbb{C}})^{ss}_{\iota^*\zeta_1}$ and $\sigma_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})^{ss}_{\zeta_1}$ are the minimal open subsets of $\mu_{\mathbb{C}}^{-1}(\iota^*\zeta_{\mathbb{C}})$ and $\sigma_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})$ containing $\mu_{\mathbb{C}}^{-1}(\iota^*\zeta_{\mathbb{C}})_{\iota^*\zeta_1}$ and $\sigma_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})_{\zeta_1}$, respectively. Then ψ gives a bijective map $\mu_{\mathbb{C}}^{-1}(\iota^*\zeta_{\mathbb{C}})^{ss}_{\iota^*\zeta_1}//H_\rho^{\mathbb{C}} \to \sigma_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})^{ss}_{\zeta_1}//H^{\mathbb{C}}$ by Propositions 4.2, 4.3, and 4.4. Moreover it is biholomorphic since $\hat{\psi}$ is obviously holomorphic and the inverse of ψ is also holomorphically defined in the proof of Proposition 4.2.

Let $(I_1^{\iota^*\zeta}, I_2^{\iota^*\zeta}, I_3^{\iota^*\zeta})$ be the hypercomplex structure on $\mu^{-1}(\iota^*\zeta)/H_\rho$ induced from (I_1, I_2, I_3) on M. Similarly, let $(I_1^\zeta, I_2^\zeta, I_3^\zeta)$ be the hypercomplex structure on $\sigma^{-1}(\zeta)/H$ induced from $(I_1 \times I_{G,1}, I_2 \times I_{G,2}, I_3 \times I_{G,3})$ on $M \times N_G$. Moreover, let $\omega_i^{\iota^*\zeta}$ and ω_i^ζ be the corresponding Kähler forms.

If H_ρ acts on $\mu^{-1}(\iota^*\zeta)$ freely, then H also acts on $\sigma^{-1}(\zeta)$ freely from Proposition 4.5, hence $\mu^{-1}(\iota^*\zeta)/H_\rho$ and $\sigma^{-1}(\zeta)/H$ become smooth hyper-Kähler manifolds by [9]. Since M and $M \times N_G$ are complete, $\mu^{-1}(\iota^*\zeta)/H_\rho$ and $\sigma^{-1}(\zeta)/H$ are complete, too. See [12] for the completeness of N_G.

The equality $\psi^*(\omega_2^\zeta + \sqrt{-1}\omega_3^\zeta) = \omega_2^{\iota^*\zeta} + \sqrt{-1}\omega_3^{\iota^*\zeta}$ follows directly from the definition of $\hat{\psi}$ in Section 4.1 and the fact that any fiber of $T^*G^{\mathbb{C}}$ are holomorphic Lagrangian submanifolds.

Next we show the corresponding of Kähler classes. For each $y \in S^2 = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3;\ y_1^2 + y_2^2 + y_3^2 = 1\}$, put

$$I_y^{\iota^*\zeta} := \sum_{i=1}^3 y_i I_i^{\iota^*\zeta}, \quad I_y^\zeta := \sum_{i=1}^3 y_i I_i^\zeta.$$

Now we take $y', y'' \in S^2$ such that $\{y, y', y''\}$ is the orthonormal basis of $\mathbb{R}^3$ with the positive orientation. Then we can apply Theorem 1.1 for the complex structure $I_y^{\iota^*\zeta}$ and I_y^ζ, and obtain a biholomorphism ψ_y. Thus we obtain a continuous family of diffeomorphisms $\{\psi_y\}_y$ parametrized by $y \in S^2$. Since S^2 is connected, the induced maps

$$\psi_y^* : H^2(\sigma^{-1}(\zeta)/H, \mathbb{R}) \to H^2(\mu^{-1}(\iota^*\zeta)/H_\rho, \mathbb{R})$$

does not depend on $y \in S^2$. Since each ψ_y identifies the holomorphic symplectic forms with respect to $I_y^{\iota^*\zeta}$ and I_y^ζ, therefore $\psi_y^*[\omega_i^\zeta] = [\omega_i^{\iota^*\zeta}]$ holds for $i = 1, 2, 3$. $\square$

Finally, we show the correspondence of the parameter spaces of two hyper-Kähler quotients.

Proposition 4.6. *Let $\zeta, \zeta' \in \mathrm{Im}\mathbb{H} \otimes Z_H$ satisfies $\iota^*\zeta = \iota^*\zeta'$. Then hyper-Kähler quotients $\sigma^{-1}(\zeta)/H$ and $\sigma^{-1}(\zeta')/H$ are canonically identified.*

Proof. Take $\zeta, \zeta' \in \mathrm{Im}\mathbb{H} \otimes Z_H$ such that $\iota^*\zeta = \iota^*\zeta'$. Then $\zeta' - \zeta \in \mathrm{Im}\mathbb{H} \otimes \mathrm{Ann}.\mathbf{h}_\rho$, and there exists a unique $A = (A_1, A_2, A_3) \in \mathrm{Im}\mathbb{H} \otimes \mathbf{g}^*$ such that $\rho^*(A, -A) = \zeta' - \zeta$ by Lemma 4.1. Define $\hat{A} \in \mathcal{N}_G$ by $\hat{A}(t) := (0, A_1, A_2, A_3)$ for all $t \in [0, 1]$. Then a C^∞ map $\sigma^{-1}(\zeta) \to \sigma^{-1}(\zeta')$ defined by $(x, [T]) \mapsto (x, [T + \hat{A}])$ gives an isomorphism $\sigma^{-1}(\zeta)/H \to \sigma^{-1}(\zeta')/H$. $\square$

5 Examples

Here we raise some examples which Theorem 1.1 can be applied to.

5.1 Hilbert schemes of k points on $\mathbb{C}^2$

Here we apply the main results obtained in the previous sections to the case of

$$M = \mathrm{End}(\mathbb{C}^k) \oplus \mathrm{End}(\mathbb{C}^k) \oplus \mathbb{C}^k \oplus (\mathbb{C}^k)^*$$

$G = U(k)$, $H = U(k) \times U(k)$ and $\rho = \mathrm{id} : H \to G \times G$. Here, H-action on M is defined by $(g_0, g_1) \cdot (A, B, p, q) := (g_0 A g_1^{-1}, g_1 B g_0^{-1}, g_0 p, q g_0^{-1})$ for $g_0, g_1 \in U(k)$, $A, B \in \mathrm{End}(\mathbb{C}^k)$, $p \in \mathbb{C}^k$ and $q \in (\mathbb{C}^k)^*$. According to [14], $Z_{H_\rho} \cong \mathbb{R}$ and $\mu^{-1}(\iota^*\zeta)/H_\rho$ is a smooth hyper-Kähler manifolds diffeomorphic to a crepant resolution of $(\mathbb{C}^2)^k/\mathcal{S}_k$ if $\iota^*\zeta \in \mathrm{Im}\mathbb{H}$ is given by $\iota^*\zeta = (t, 0, 0)$ for $t \neq 0$ in this situation. Here, $\mathcal{S}_k$ is the symmetric group acting on $(\mathbb{C}^2)^k$. If $\iota^*\zeta = 0$, then $\mu^{-1}(0)/H_\rho$ is isometric to $(\mathbb{C}^2)^k/\mathcal{S}_k$ with Euclidean metric.

Then we have a family of smooth hyper-Kähler manifolds $\sigma^{-1}(\zeta)/H$ which are biholomorphic to $\mu^{-1}(\iota^*\zeta)/H_\rho$. In particular, we can study $\sigma^{-1}(0)/H$ which gives a singular hyper-Kähler metric on $(\mathbb{C}^2)^k/\mathcal{S}_k$ as follows.

Theorem 5.1. *Let M, H, G, ρ be as above. Then $\sigma^{-1}(0)/H$ is isometric to $(\mathbb{C}^2_{Taub-NUT})^k/\mathcal{S}_k$ on their regular parts, where $\mathbb{C}^2_{Taub-NUT}$ is Taub-NUT space.*

Before the proof of Theorem 5.1, we see that N_L is identified with the open subset of $L \times \mathbf{l}^3$ as follows by [3], for any compact Lie group L. Let $T \in \mathcal{N}_L$ and $f : [0,1] \to L$ be the solution of the initial value problem

$$\begin{aligned} \mathrm{Ad}_{f(s)}T_0(s) + f(s)\frac{d}{ds}f(s)^{-1} &= 0, \\ f(1) &= 1. \end{aligned}$$

Then a C^∞ map $\phi : N_L \to L \times \mathbf{l}^3$ is defined by

$$\phi([T]) := (f(0)^{-1}, T_1(1), T_2(1), T_3(1)).$$

ϕ is an diffeomorphism from N_L to an open subset of $L \times \mathbf{l}^3$. In particular, ϕ is surjective and an isomorphism of hyper-Kähler manifolds if L is a torus, therefore we may assume $N_{T^k} = T^k \times \mathbb{R}^{3k}$.

Next we begin the proof of Theorem 5.1. The inclusion $T^k \subset U(k)$ which is given by

$$T^k = \{\mathrm{diag}(g_1, \cdots, g_k) \in U(k);\ g_1, \cdots, g_k \in S^1\}$$

induces an embedding $N_{T^k} \subset N_{U(k)}$. Now we put

$$\begin{aligned} M_0 &:= \{(A, B, 0, 0) \in M;\ A = \mathrm{diag}(a_1, \cdots, a_k),\ B = \mathrm{diag}(b_1, \cdots, b_k)\} \\ &\cong \mathbb{C}^k \oplus \mathbb{C}^k, \end{aligned}$$

then $\hat{M}_0 := M_0 \times N_{T^k}$ is a hyper-Kähler submanifold of $\hat{M} := M \times N_{U(k)}$.

Let a closed sub group $H_0 \subset H$ be generated by

$$\{(g\chi, \chi) \in U(k) \times U(k);\ g \in T^k,\ \chi \in \mathcal{S}_k\},$$

then H_0 is isomorphic to $\mathcal{S}_k \ltimes T^k$. Then, H_0-action is closed on $\hat{M}_0$, and we obtain the hyper-Kähler moment map $\sigma_0 := \iota_0^* \circ \sigma|_{\hat{M}_0} : \hat{M}_0 \to \mathrm{Im}\mathbb{H} \otimes \mathbf{h}_0^*$, where $\iota_0^* : \mathbf{h}^* \to \mathbf{h}_0^*$ is the adjoint map of the inclusion $\mathbf{h}_0 \hookrightarrow \mathbf{h}$. Here, $\mathbf{h}_0 = \mathbf{u}(k) \oplus \{0\}$ is the Lie algebra of H_0.

Lemma 5.2. *We have $\sigma_0^{-1}(0) = (\sigma|_{\hat{M}_0})^{-1}(0)$, and the naturally induced map $\sigma_0^{-1}(0)/H_0 \to \sigma^{-1}(0)/H$ is injective.*

Proof. For $x = (A, B, 0, 0) \in M$, we have

$$\begin{aligned}
\hat{\mu}(x) &= (\sqrt{-1}(B^*B - AA^*), \sqrt{-1}(-AB - B^*A^*), -AB + B^*A^*) \\
&\quad \oplus(\sqrt{-1}(A^*A - BB^*), \sqrt{-1}(BA + A^*B^*), BA - A^*B^*) \\
&\in \mathrm{Im}\mathbb{H} \otimes (\mathbf{u}(k) \oplus \mathbf{u}(k)),
\end{aligned}$$

where $\mathbf{u}(k)$ is the Lie algebra of $U(k)$, and we identify $\mathbf{u}(k) \cong \mathbf{u}(k)^*$ by the bilinear form $(u, v) \mapsto \mathrm{tr}(uv^*)$. If $x \in M_0$, we can put

$$A = \mathrm{diag}(a_1, \cdots, a_k),\ B = \mathrm{diag}(b_1, \cdots, b_k),$$

and we obtain

$$\hat{\mu}(x) = -\sqrt{-1}\tau(x) \oplus \sqrt{-1}\tau(x) \in \mathrm{Im}\mathbb{H} \otimes (\mathbf{t}^k \oplus \mathbf{t}^k),$$

where $\mathbf{t}^k := \mathrm{Lie}(T^k) \subset \mathbf{u}(k)$, and $\tau = (\tau_1, \tau_2, \tau_3) : M_0 \to \mathrm{Im}\mathbb{H} \otimes \mathbb{R}^k$ is the hyper-Kähler moment map with respect to the tri-Hamiltonian T^k-action on M_0 defined by

$$\begin{aligned}
\tau_1(x) &= \mathrm{diag}(|a_1|^2 - |b_1|^2, \cdots, |a_k|^2 - |b_k|^2), \\
\tau_2(x) &= \mathrm{diag}(2\mathrm{Re}(a_1 b_1), \cdots, 2\mathrm{Re}(a_k b_k)), \\
\tau_3(x) &= \mathrm{diag}(2\mathrm{Im}(a_1 b_1), \cdots, 2\mathrm{Im}(a_k b_k)).
\end{aligned}$$

Under the identification $(\theta, y) \in T^k \times \mathbb{R}^{3k} = N_{T^k}$, we obtain $\rho^*(\nu(\theta, y)) = \sqrt{-1}(y, -y)$. Thus we have $\sigma(x, \theta, y) = \sqrt{-1}(-\tau(x) + y, \tau(x) - y)$ and $\sigma_0(x, \theta, y) = \sqrt{-1}(-\tau(x) + y)$ for $(x, t, y) \in \hat{M}_0$, which implies $\sigma_0^{-1}(0) = (\sigma|_{\hat{M}_0})^{-1}(0)$. Then we obtain $\sigma_0^{-1}(0)/H_0 \to \sigma^{-1}(0)/H$ by the inclusion $\sigma_0^{-1}(0) \subset \sigma^{-1}(0)$.

Next we show the injectivity of $\sigma_0^{-1}(0)/H_0 \to \sigma^{-1}(0)/H$. Let $(x, \theta, y) \in \sigma_0^{-1}(0)$ and $(g_0, g_1) \in H = U(k) \times U(k)$ satisfy $(g_0, g_1) \cdot (x, \theta, y) \in \sigma_0^{-1}(0)$. Since $(g_0, g_1)\cdot(x, \theta, y) = ((g_0, g_1)\cdot x, g_0\theta g_1^{-1}, \mathrm{Ad}_{g_1} y)$, we have $\tilde{\theta} := g_0\theta g_1^{-1} \in T^k$. For $x = (A, B, 0, 0)$,

$$\begin{aligned}
(g_0, g_1)x &= (g_0 A g_1^{-1}, g_1 B g_0^{-1}, 0, 0) \\
&= (g_0 A \theta^{-1} g_0^{-1}\tilde{\theta}, \tilde{\theta}^{-1} g_0 \theta B g_0^{-1}, 0, 0) =: (\tilde{A}, \tilde{B}, 0, 0) \in M_0,
\end{aligned}$$

then we have equalities between diagonal matrices $g_0 A\theta^{-1} g_0^{-1} = \tilde{A}\tilde{\theta}^{-1}$ and $g_0\theta B g_0^{-1} = \tilde{\theta}\tilde{B}$. By comparing the eigenvalues of both sides, we can see there exist $\chi \in \mathcal{S}_k$ such that $\tilde{a}_{\chi(i)}\tilde{\theta}_{\chi(i)}^{-1} = a_i\theta_i^{-1}$ and $\tilde{b}_{\chi(i)}\tilde{\theta}_{\chi(i)} = b_i\theta_i$ for $i = 1, \cdots, k$, where $\tilde{a}_i, \tilde{b}_i, \theta_i, \tilde{\theta}_i$ are the $i \times i$ components of $\tilde{A}, \tilde{B}, \theta, \tilde{\theta}$, respectively. This implies that (x, t, y) and $(g_0, g_1) \cdot (x, \theta, y)$ lie on the same H_0-orbit, since $y = \tau(x)$. $\square$

Proof of Theorem 5.1. It is easy to see that $\sigma_0^{-1}(0)/H_0 \to \sigma^{-1}(0)/H$ preserves the hyper-Kähler structures. Since $\sigma_0^{-1}(0)/H_0 = (\sigma_0^{-1}(0)/T^k)/\mathcal{S}_k$ and $\sigma_0^{-1}(0)/T^k$ is isomorphic to $(\mathbb{C}^2_{Taub-NUT})^k$, therefore $\sigma^{-1}(0)/H$ contains $(\mathbb{C}^2_{Taub-NUT})^k/\mathcal{S}_k$ as a hyper-Kähler suborbifold. From Theorem 1.1, the quotient space $\sigma^{-1}(0)/H$ is homeomorphic to $\mu^{-1}(0)/H_\rho$, which is $(\mathbb{C}^2)^k/\mathcal{S}_k$ by [14]. Since $(\mathbb{C}^2)^k/\mathcal{S}_k$ is connected, and $(\mathbb{C}^2_{Taub-NUT})^k/\mathcal{S}_k$ is complete, the embedding $\sigma_0^{-1}(0)/H_0 \to \sigma^{-1}(0)/H$ should be isomorphic. □

5.2 Quiver varieties

The setting considered in Section 5.1 can be generalized to quiver varieties defined by Nakajima [15], which contains ALE spaces constructed by [11]. Quiver varieties are constructed as hyper-Kähler quotient as follows.

Let $Q = (V, E, s, t)$ be a finite oriented graph, that is, V and E are finite sets with maps $s, t : E \to V$, where $s(h) \in V$ is a source of a quiver $h \in E$, and $t(h) \in V$ is a target. More over E is decomposed into $E = \Omega \sqcup \overline{\Omega}$, with one to one correspondence $\Omega \to \overline{\Omega}$ denoted by $h \mapsto \bar{h}$ satisfying $s(\bar{h}) = t(h)$ and $t(\bar{h}) = s(h)$. Next we fix a dimension vector $v = (v_k)_{k\in V}$, where each v_k is a positive integer. Then the action of $\prod_{k\in V} U(v_k)$ on

$$M = \bigoplus_{h\in\Omega} \mathrm{Hom}(\mathbb{C}^{v_{s(h)}}, \mathbb{C}^{v_{t(h)}}) \oplus \bigoplus_{h\in\Omega} \mathrm{Hom}(\mathbb{C}^{v_{t(h)}}, \mathbb{C}^{v_{s(h)}}) \tag{7}$$

by $(g_k)_k \cdot (A_h, B_h)_h := (g_{t(h)} A_h g_{s(h)}^{-1}, g_{s(h)} B_h g_{t(h)}^{-1})_h$. Then the quiver varieties are constructed by taking hyper-Kähler quotients for this situation.

Here we explain the settings of Taub-NUT deformations for quiver varieties, which contain the case of [2]. Let M be as (7). We define H, G, ρ as follows so that $H_\rho = \prod_{k\in V} U(v_k)$. We take another finite oriented graph $\tilde{Q} = (\tilde{V}, E, \tilde{s}, \tilde{t})$ with a surjection $\pi : \tilde{V} \to V$ satisfying $\pi(\tilde{s}(h)) = s(h)$ and $\pi(\tilde{t}(h)) = t(h)$ for all $h \in H$. We label elements of $\pi^{-1}(k)$ numbers as $\pi^{-1}(k) = \{k_1, k_2, \cdots, k_{N_k}\}$. Note that $\tilde{Q}$ may be disconnected even if Q is a connected graph. A dimension vector $v' = (v_{\tilde{k}})_{\tilde{k}\in\tilde{V}}$ is determined by $v_{\tilde{k}} = v_{\pi\tilde{k}}$ for all $\tilde{k} \in \tilde{V}$. Then we define $H := \prod_{\tilde{k}\in\tilde{V}} U(v_{\tilde{k}})$ and $G := \prod_{k\in V'} U(v_k)^{N_k - 1}$, where $V' = \{k \in V;\ \sharp\pi^{-1}(k) \geq 2\}$. A homomorphism $\rho : H \to G \times G$ is defined by

$$\rho((g_{\tilde{k}})_{\tilde{k}\in\tilde{V}}) = ((g_{k_1}, g_{k_2}, \cdots, g_{k_{N_k - 1}}), (g_{k_2}, \cdots, g_{k_{N_k - 1}}, g_{k_{N_k}}))_{k\in V'}$$

Then $\mu^{-1}(\iota^*\zeta)/H_\rho$ becomes a quiver variety, and we obtain another hyper-Kähler quotient $\sigma^{-1}(\zeta)/H$ diffeomorphic to $\mu^{-1}(\iota^*\zeta)/H_\rho$.

5.3 Toric hyper-Kähler varieties

In the previous sections we assumed that H and G are compact. However, the compactness is not essential for the proof of Theorem 1.1, we need only Ad_G-invariant positive definite inner products on its Lie algebra and the existence of hyper-Kähler metrics on N_G with tri-Hamiltonian $G\times G$-actions. In this subsection we consider the case of noncompact abelian Lie groups $H = \mathbb{R}^N$ and $G = \mathbb{R}^N/\mathbf{k}$, where the vector subspace $\mathbf{k} \subset \mathbb{R}^N$ is given by $\mathbf{k} = \mathbf{k}_{\mathbb{Z}} \otimes \mathbb{R}$ for some submodule $\mathbf{k}_{\mathbb{Z}} \in \mathbb{Z}^N$. $\rho : H \to G \times G$ is defined by $\rho(v) := (v \text{ mod. } \mathbf{k}, 0)$, then $\bar{\rho}$ defined by (1) is surjective. In this case we put $N_G := G \times G \times G \times G$ with the Euclidean metric, and $G \times G$-action on N_G is defined by $(g_0, g_1) \cdot (h_0, h_1, h_2, h_3) := (h_0 + g_0 - g_1, h_1, h_2, h_3)$. Then Theorem 2.1, 2.2 and Proposition 2.5 hold in this case. Let $M = \mathbb{H}^N$, and define H-action on M by

$$(t_1, \cdots, t_N) \cdot (x_1, \cdots, x_N) := (x_1 e^{-2\pi i t_1}, \cdots, x_N e^{-2\pi i t_N}).$$

The the hyper-Kähler quotient $\mu^{-1}(\iota^*\zeta)/H_\rho$ becomes a toric hyper-Kähler variety, and $\sigma^{-1}(\zeta)/H$ is its Taub-NUT deformation defined in [1]. Theorem 1.1 can be also applied to this situation.

5.4 Hyper-Kähler manifolds with tri-Hamiltonian actions

Here we show that the limited case of Theorem 7 of [4] also follows from Theorem 1.1. Let $M = \mathbb{H}^N$ and $H \subset Sp(N)$. Take a normal closed subgroup $H_\rho \subset H$ and put $G := H/H_\rho$. Let $\rho : H \to G \times G$ be given by $\rho(h) := (1, hH_\rho)$. Then $X = \mu^{-1}(\iota^*\zeta)/H_\rho$ is a hyper-Kähler manifolds with tri-Hamiltonian G-action, and $\sigma^{-1}(\zeta)/H$ is the *modification* of $\mu^{-1}(\iota^*\zeta)/H_\rho$ defined in Section 5 of [4]. From Theorem 1.1, we have the following results.

Theorem 5.3. *Let $X = \mu^{-1}(\iota^*\zeta)/H_\rho$ be a tri-Hamiltonian G hyper-Kähler manifold defined as above. Then the modification of X in the sense of Section 5 of [4] by the tri-Hamiltonian G-action is isomorphic to X as holomorphic symplectic manifolds, hence diffeomorphic.*

By Theorem 7 of [4], we have already known that $\sigma^{-1}(\zeta)/H$ is diffeomorphic to $\hat{\mu}^{-1}(\nu^1(N_G) + \zeta)/H_\rho$, which is an open subset of X. Theorem 5.3 asserts that this open subset is diffeomorphic to X, even if it is a proper subset of X.

Acknowledgment. The author was supported by Grant-in-Aid for JSPS Fellows $(23 \cdot 1432)$

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